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THE UNIVERSITY OF ALBERTA

PLASTIC ANISOTROPY

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Plastic Anisotropy" submitted by Bruce Alexander Slevinsky in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

Classical plasticity theory for macroscopically isotropic materials is briefly outlined.

An extension of this theory to anisotropic materials, as developed by Hill, is discussed.

The elastic response of hexagonal single crystals is investigated and a speculative analogy between the yield function and the incompressible crystal strain energy function is proposed. Yield and plastic deformation under hydrostatic pressure are considered qualitatively for single crystals and polycrystalline aggregates.

The problem of plastic torsion of a thin walled tube is studied and relations are developed for the orientation of the principal axes of the deformation and the variation of the anisotropic parameters. Results of the analysis are compared with an analysis of the same problem by Bailey, et al.*

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*Bailey, J. A.; Haas, S. L. and K. L. Nawab: Anisotropy in Plastic Torsion. Journal of Basic Engineering, March 1972, p. 231.

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CHAPTER 1

Classical Plasticity

1.1 Introduction

The theory of plasticity, at least for macroscopically isotropic materials, is a "classical" theory in the same sense as the classical theory of elasticity. It has been applied to many problems involving metal working processes, limit design, and analysis of standard mechanical tests and it has a considerable predictive value. For materials which are macroscopically anisotropic or in which anisotropy develops with deformation, the theory is less well developed and there are relatively few complete solutions of technologically important problems. Most solutions which do exist, however, require that the material be initially isotropic. Many similarities exist between the theories for isotropic and anisotropic materials. Therefore, before beginning a discussion of plastic anisotropy it is desirable to discuss the basic concepts of the classical theory of plasticity for isotropic materials.

1.2 The Classical Theory of Plasticity for Isotropic Materials

The classical theory of plasticity was developed from observations of the macroscopic behaviour of plastic solids under uniform states of combined stress. Since data for metals (and particularly cubic metals) have predominated, the general theory is related specifically to their properties; although the plastic deformation of other materials may be described adequately by the theory. The theory is based on the assumptions that:

a) the material is a continuum.

b) creep and thermal phenomena may be neglected (i.e., temperatures are assumed to be much lower than the melting point of the material).

c) there is no hysteresis loop in elastic loading and unloading.

d) there is no size effect.

One further restriction on the classical theory is that the elastic strains are small (of the order of Y/E). This limits the theory to moderate hydrostatic stresses.

The basic observation of an ideal metal undergoing a cycle of plastic deformation and loading is as follows. Initially, as the body is loaded, it responds elastically with the stress and strain linearly related. Under some critical loading condition the material yields or begins to deform plastically, and departs from the original linear stress-strain relationship. As the strain increases so too does the stress (the material is said to work harden in this case). On unloading the material responds elastically and linearly so that the slope of the stress-strain curve is the same as in the initial stage of loading. The elastic constants are assumed to be unchanged by the deformation as long as they are defined with respect to the current plastically deformed state of the body. During the deformation, the body undergoes no permanent plastic volume change. A further observation is that the yielding is not influenced by a moderate hydrostatic stress (i.e., small compared to the bulk modulus).

The mathematical relation which defines the limit of elasticity under the system of combined stresses is called the yield criterion, and

for the isotropic case it is a function.

$$f(J_1,J_2,J_3) = 0 \tag{1.2.1}$$

where J_1,J_2,J_3 are the three basic invariants of the stress tensor $\underset{\sim}{\sigma}$. The requirement that moderate hydrostatic stresses not influence yielding and the assumption that there is no Bauschinger effect further imply that the yield function is of the form

$$f(J_2',J_3') = 0 \tag{1.2.2}$$

where f is an even function of J_3' and J_1',J_2',J_3' are the three basic invariants of the stress tensor $\underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - \frac{1}{3}tr(\underset{\sim}{\sigma})\,\underset{\sim}{I}$.

There have been many proposals for yield criteria, but the forms which are most useful are those due to Tresca and von Mises.

The Tresca yield criterion states that the material yields when the maximum shear stress reaches a critical value. The yield function is

$$\sigma_{max} - \sigma_{min} = 2k \qquad \text{where } k = Y/2$$

and does not involve the intermediate normal and is unaffected by the pressure. The von Mises yield criterion states that yielding occurs when J_2' reaches a critical value and is not dependent on J_3' at all. The yield function is

$$\begin{aligned} f(\sigma_{ij}) &= (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 6k^2 \\ &= (\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2) \\ &= 6k^2 \end{aligned} \qquad *(1.2.3)$$

where $k = Y/\sqrt{3}$ is dependent on the strain history. Physically, the von Mises yield criterion can be interpretted to imply that yielding occurs when:

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*The subscripts 1, 2, 3, refer to the principal stress system.

a) the elastic distortional strain energy per unit volume reaches a critical value.

b) the octahedral shear stress reaches a critical value ($\tau oct = \sqrt{\frac{2}{3}}\, k$).

The von Mises criterion fits most data better than does the Tresca criterion, although, it is more difficult to handle mathematically in some problems. The Tresca criterion is often used as a piecewise linear approximation to the von Mises yield criterion.

There is no one-to-one stress-plastic strain relationship in plastic deformation; the current stress and plastic strain increment are related. It has been deduced {1}* that this relation is the form

$$d\varepsilon ij^{p} = \frac{h\partial g}{\partial \sigma ij}\, df \qquad (1.2.4)$$

where g and h are scalar functions of the stress invariants J_2' and J_3' and the strain history. This form of the flow rule predicts that there is no plastic volume change and the principal axes of the plastic strain-increment tensor and the stress tensor are coaxial. When g = f (or some multiple or power of f) certain variational principles and uniqueness theorems are valid. These are necessary for the development of the mathematical theory of plasticity. Bishop and Hill {2} have shown that this relationship is necessarily true for isotropic aggregates of cubic crystals.

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*The numbers in parenthesis refer to references listed in the Bibliography at the end of this thesis.

1.3 Finite Elastic-Plastic Theory

Classical plasticity theory as just outlined is restricted to moderate hydrostatic stresses. When pressures are large (approaching several percent of the bulk modulus) there is a finite elastic volume change which influences the yielding behaviour of the material. Haddow and Hrudey {3} have shown that, if Drucker's hypothesis {4} is valid the yield function is of the form

$$g\ (\underset{\sim}{\tau}, \theta) = 0 \qquad (1.3.1)$$

where $\underset{\sim}{\tau} = \underset{\sim}{\sigma}\beta^{2/3}$ and $\beta = \rho^1/\rho$ is the ratio of densities in unstressed and stressed states; θ = temperature. Further, if there is no plastic volume change, the yield function for isotropic materials is of the form

$$g\ (I_2', I_3', \theta) = 0 \qquad (1.3.2)$$

where $\underset{\sim}{\tau}' = \underset{\sim}{\tau} - 1/3 tr\ (\underset{\sim}{\tau})\ \underset{\sim}{I}$ and I_1', I_2', I_3' are the three basic invariants of $\underset{\sim}{\tau}'$. The explicit form of the yield function has not been investigated, but if it maintains the same form as that due to von Mises

$\tau ij' \ \tau ij' = 2k^2$ (where the usual summation convention for tensors is applied)

then the von Mises yield function is a limiting case for zero pressure. The flow rule is of the form

$$\underset{\sim}{D\pi} = \gamma \frac{\partial g}{\partial \underset{\sim}{\tau}} \qquad (1.3.3)$$

where $\underset{\sim}{D\pi}$ is the properly invariant form of the plastic rate of deformation tensor that corresponds to the yield state of stress ($g(\underset{\sim}{\tau}', \theta) = 0$).

1.4 Conclusion

This has, of necessity, been a very brief outline of the classical theory of plasticity of isotropic materials and is meant to deal only with some of the major points of the theory. It has left

out development of the variational principles and proof of the uniqueness theorem as well as the development of methods of solution of special problems in plasticity. Further information can be obtained from Hill's book {5} especially the first three chapters.

CHAPTER 2

Plastic Anisotropy

2.1 Introduction

Whether as a result of microscopic deformation processes or the original structure of the poly crystalline aggregate, anisotropy exists or develops in most processes involving plastic deformation of metals. The classical theory previously outlined is incapable of describing accurately the behaviour in such cases. To construct a theory which will deal with plastic anisotropy it is necessary to modify the isotropic theory. The anisotropic theory requires a yield criterion and flow rule which are directionally dependent and which are capable of changing as the state of anisotropy of the element changes. One further condition placed on the theory is that for vanishingly small anisotropy, the yield criterion and stress-strain increment relation should approach those of the isotropic case.

2.2 Yield Criterion

The theory as developed here is set up to consider states of anisotropy with three mutually orthogonal planes of symmetry at every point. The principal axes of anisotropy, defined by the intersections of the planes of symmetry, may vary in direction throughout the body and are used as local Cartesian axes of reference for elements of the body. Following the form given by Green and Naghdi {6} the yield criterion is assumed to be of the form

$$f\ (\sigma_{kl},\ e_{kl}^{P}) = 0 \qquad (2.2.1)$$

where σ_{kl} and e_{kl}^{P} are the stresses and plastic strains respectively referred to some set of axes in the body.

Although not specifically stated in Green and Naghdi it is assumed here that the stresses and plastic strains are referred to the principal axes of anisotropy of the element.

Following the development due to Hill, {7}, the explicit form of the yield criterion is assumed to be a quadratic function in the stress components which reduces to von Mises yield criterion for vanishingly small anisotropy. The yield criterion is of the form

$$2f\,(\sigma ij) = F(\sigma y-\sigma z)^2 + G(\sigma z-\sigma x)^2 + H(\sigma x-\sigma y)^2$$

$$2L\tau yz^2 + 2M\tau zx^2 + 2N\tau xy^2 = 1 \qquad (2.2.2)$$

where F, G, H, L, M, N are parameters characteristic of the current

state of anisotropy.

The yield criterion is this form implies

a) there is no Bauschinger effect.

b) moderate superimposed hydrostatic loads do not influence yielding.

If X, Y, Z and R, S, T are the tensile yield stresses and the shear yield stresses respectively referred to the principal axes of anisotropy, then

$$\frac{1}{X^2} = G + H, \quad 2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}, \quad 2L = \frac{1}{R^2}$$

$$\frac{1}{Y^2} = H + F, \quad 2G = \frac{1}{X^2} + \frac{1}{Z^2} - \frac{1}{Y^2}, \quad 2M = \frac{1}{S^2} \qquad (2.2.3)$$

$$\frac{1}{Z^2} = F + G, \quad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}, \quad 2N = \frac{1}{T^2}$$

Clearly only one of F, G, H can be negative and L, M, N are essentially positive. For isotropy

$$L = M = N = 3F = 3G = 3H \qquad (2.2.4)$$

And the yield criterion (2.2.2) reduces to von Mises' criterion with

$$2F = 1/Y^2$$

A full description of the state of anisotropy requires a knowledge of the orientation of the principal axes of anisotropy as well as the six parameters F, G, H, L, M, N (or the yield stresses X, Y, Z, R, S, T).

2.3 Stress-Plastic Strain Increment Relation

More by analogy with the isotropic case than by any physical or theoretical justification, a plastic potential flow rule is assumed with the potential function and yield function assumed to be equal. This leads directly to the stress-strain increment relations, referred to the principal axes of anisotropy.

$$\begin{aligned} d\varepsilon x &= d\lambda\,\{H(\sigma x-\sigma y) + G(\sigma x-\sigma z)\}, \quad d\varepsilon yz = d\lambda\, L\tau yz, \\ d\varepsilon y &= d\lambda\,\{F(\sigma y-\sigma z) + H(\sigma y-\sigma x)\}, \quad d\varepsilon zx = d\lambda M\tau zx, \qquad (2.3.1) \\ d\varepsilon z &= d\lambda\{G(\sigma z-\sigma x) + F(\sigma x-\sigma z)\}, \quad d\varepsilon xy = d\lambda N\tau xy, \end{aligned}$$

Since the yield criterion is not pressure dependent, the plastic potential flow rule with the plastic potential indentical to the yield function predicts no plastic volume change. Note that reversing the signs of all of the stresses in a given stress system reverses the direction of the plastic strain increments. The principal axes of stress and strain-increment are not coaxial unless the principal axes of anisotropy and stress coincide.

CHAPTER 3

Some Microscopic Aspects of the Theory

3.1 Introduction

So far the theory has been presented from a macroscopic viewpoint with no attempt to correlate macroscopic observations with the properties of single crystals. Bishop and Hill {8}, {9} and Hill {10} have investigated the plastic deformation of a polycrystalline aggregate under the assumption that the only active means of deformation is shear along preferred orientations in preferred planes. They showed that for cubic crystals (and particularly face-centred cubic crystals) the yield function and plastic potential were identical. The same analysis is not applicable to non-cubic crystals where additional deformation modes are required to produce a general deformation. Several important aspects of the theory, however, can be investigated from the basis of the elastic response of single non-cubic crystals and their polycrystalline aggregates. Hexagonal crystals and the more general trigonal group, are the non-cubic crystals which are studied in the following work.

Under pressure, hexagonal crystals exhibit both volume change and deviatoric strain; this behaviour poses the question of whether yielding and/or plastic deformation are possible under the action of pressure alone. There is also the question of yielding of a polycrystalline hexagonal aggregate under pressure due to the interaction between the individual crystals.

The generalized form of Hooke's Law is

$$\varepsilon_{ij} = S_{ijkl} \ \sigma_{kl} \qquad (\text{or } \underset{\sim}{e} = \underset{\sim}{S} \ \underset{\sim}{\sigma}) \qquad (3.1.1)$$

with the inverse relation

$$\sigma_{ij} = C_{ijkl}\, e_{kl} \qquad (\text{or } \underset{\sim}{\sigma} = \underset{\approx}{C}\underset{\sim}{e}) \qquad (3.1.2)$$

where $\underset{\sim}{S}$ and $\underset{\sim}{C}$ are the fourth order compliance and stiffness tensors. The symmetry of the Sijkl and Cijkl in the first two and last two pairs of suffixes, as implied by the symmetry of the stress and strain tensors, makes it possible to use the matrix notation proposed by Nye {11}, summarized by

tensor notation	11	22	33	12,21	13,31	23,32
matrix notation	1	2	3	6	5	4

This makes it possible to represent the stress and strain systems by six-component vectors and the stiffness and compliance tensors by 6 x 6 matrices. The vectors and matrices, however, are no longer tensors and do not obey the tensor transformation law. They will be represented in this paper by the symbols

[σ] for the stress vector.

[e] for the strain vector.

[S] for the compliance matrix.

and [C] for the stiffness matrix.

The stress and strain systems in the matrix form are

$$[\sigma] = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad \text{and} \qquad [e] = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

where $\sigma_1=\sigma_{11}, \sigma_2=\sigma_{22}, \sigma_3=\sigma_{33}, \sigma_4=\sigma_{23}, \sigma_5=\sigma_{31}, \sigma_6=\sigma_{12}$

$e_1 = e_{11}, e_2=e_{22}, e_3=e_{33}, e_4=2e_{23}, e_5=2e_{31}, e_6=2e_{12}$

The matrix form of Hooke's Law is

$$[e] = [S][\sigma] \tag{3.1.3}$$

with the inverse relation

$$[\sigma] = [C]\ [e] \tag{3.1.4}$$

3.2 Hexagonal Single Crystals

The compliance and stiffness matrices for hexagonal crystals are

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{12} & S_{11} & S_{13} & & & \\ S_{13} & S_{13} & S_{33} & & & \\ & & & S_{44} & & \\ & & & & S_{44} & \\ & & & & & 2(S_{11}-S_{12}), \end{bmatrix}$$

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & 1/2\ (C_{11}-C_{12}) \end{bmatrix}$$

From Hooke's Law the response to a general stress system is

$$[e] = [S]\ [\sigma]$$

which for the hexagonal single crystal becomes

$$[e] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{12} & S_{11} & S_{13} & & & \\ S_{13} & S_{13} & S_{33} & & & \\ & & & S_{44} & & \\ & & & & S_{44} & \\ & & & & & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

$$= \begin{bmatrix} S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 \\ S_{12}\sigma_1 + S_{11}\sigma_2 + S_{13}\sigma_3 \\ S_{13}\sigma_1 + S_{13}\sigma_2 + S_{33}\sigma_3 \\ S_{44}\sigma_4 \\ S_{44}\sigma_5 \\ 2(S_{11}-S_{12})\ \sigma_6 \end{bmatrix}$$

For a hydrostatic stress the response is

$$[e] = P\begin{bmatrix} S_{11} + S_{12} + S_{13} \\ S_{12} + S_{11} + S_{13} \\ S_{13} + S_{13} + S_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which has the hydrostatic and deviatoric parts

$$[e]_H = \frac{P}{3}\begin{bmatrix} 2S_{11} + 2S_{12} + 4S_{13} + S_{33} \\ 2S_{11} + 2S_{12} + 4S_{13} + S_{33} \\ 2S_{11} + 2S_{12} + 4S_{13} + S_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$[e]_D = \frac{P}{3}\begin{bmatrix} S_{11} + S_{12} - S_{13} - S_{33} \\ S_{11} + S_{12} - S_{13} - S_{33} \\ -2(S_{11} + S_{12} - S_{13} - S_{33}) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $[e]_H$ and $[e]_D$ are the matrix notation forms of the tensors $tr(\underset{\sim}{e})\ \underset{\sim}{I}$ and $\underset{\sim}{e}' = \underset{\sim}{e} - 1/3tr(\underset{\sim}{e})\ \underset{\sim}{I}$ respectively.

For a deviatoric stress the response is

$$[e] = \begin{bmatrix} \sigma_1\ (S_{11}-S_{13}) + \sigma_2(S_{12}-S_{13}) \\ \sigma_1\ (S_{12}-S_{13}) + \sigma_2(S_{11}-S_{13}) \\ \sigma_1\ (S_{13}-S_{33}) + \sigma_2(S_{13}-S_{33}) \\ S_{44}\sigma_4 \\ S_{44}\sigma_5 \\ 2(S_{11}-S_{12})\sigma_6 \end{bmatrix}$$

which has the hydrostatic and deviatoric parts

$$[e]_H = \frac{(\sigma_1+\sigma_2)}{3}\begin{bmatrix} S_{11} + S_{12} - S_{13} - S_{33} \\ S_{11} + S_{12} - S_{13} - S_{33} \\ S_{11} + S_{12} - S_{13} - S_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$[e]_D = \frac{1}{3}\begin{bmatrix} \sigma_1(2S_{11}-S_{12}-2S_{13}+S_{33}) + \sigma_2(-S_{11}+2S_{12}-2S_{13}+S_{33}) \\ \sigma_1(-S_{11}+2S_{12}-2S_{13}+S_{33}) + \sigma_2(2S_{11}-S_{12}-2S_{13}+S_{33}) \\ \sigma_1(-S_{11}-S_{12}+4S_{13}-2S_{33}) + \sigma_2(-S_{11}-S_{12}+4S_{13}-2S_{33}) \\ 3S_{44}\sigma_4 \\ 3S_{44}\sigma_5 \\ 6(S_{11}-S_{12})\sigma_6 \end{bmatrix}$$

Note that both hydrostatic and deviatoric forms of loading produce volume change as well as distortion. The values of the coupling constant $(S_{11}+S_{12}-S_{13}-S_{33})$ and the volume change constant $(2S_{11}+2S_{12}+4S_{13}+S_{33})$ are tabulated for a variety of hexagonal and trigonal materials in Table 3.1. The values are calculated from data of the elastic parameters in the book by Simmons and Wang {12}, except for Tellurium which uses data from Schmid and Boas {13}. Although not explicitly shown here, the response of trigonal crystals to hydrostatic stress is the same as for hexagonal crystals.

3.3 Stored Energy Function for Hexagonal Crystals

The stored energy per unit volume, W, is given by

$$W = \frac{1}{2}[\sigma]^T [S] [\sigma] \tag{3.3.1}$$

$$= \frac{1}{2}[\sigma]^T [e]$$

$$= \frac{1}{2}\begin{bmatrix}\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6\end{bmatrix} \begin{bmatrix} S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 \\ S_{12}\sigma_1 + S_{11}\sigma_2 + S_{13}\sigma_3 \\ S_{13}\sigma_1 + S_{13}\sigma_2 + S_{33}\sigma_3 \\ S_{44}\sigma_4 \\ S_{44}\sigma_5 \\ 2(S_{11}-S_{12})\sigma_6 \end{bmatrix}$$

$$= \frac{1}{2}\quad S_{11}(\sigma_1^2+\sigma_2^2+2\sigma_6^2) + S_{12}(2\sigma_1\sigma_2-2\sigma_6^2) + S_{13}(2\sigma_1\sigma_2+2\sigma_2\sigma_3)$$

$$+ S_{33}(\sigma_3^2) + S_{44}(\sigma_4^2+\sigma_5^2)$$

Using $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}' + \frac{1}{3}tr(\underset{\sim}{\sigma})\underset{\sim}{I}$, this becomes (where $P = \frac{1}{3}tr(\underset{\sim}{\sigma})$)

TABLE 3.1

Coupling and Volume Change Constants For Selected Hexagonal and Trigonal Materials

Material	$S_{11}+S_{12}-S_{13}-S_{33}$ (10^{-12} cm^2/dyne)	$2S_{11}+2S_{12}+4S_{13}+S_{33}$ (10^{-12} cm^2/dyne)
Be (300°K)	.0294	0.2754
Cd (300°K)	-1.3747	2.0734
αCo	-0.0081	0.5271
Hf	-0.0112	0.9206
Mg	-0.0525	2.8311
Re	-0.0004	0.2728
Te	3.22	5.14
Ti	-0.0129	0.9324
αTl (300°K)	0.2292	2.8103
Y	-0.1145	2.4200
Zn	-1.2208	1.6921
†Calcite (0kb)	-0.5629	1.3975
†Calcite (1kb)	-0.5644	1.3933
†Calcite (2kb)	-0.5666	1.3871
†Calcite (3kb)	-0.5657	1.3790
†Calcite (4kb)	-0.5728	1.3660
†Calcite (5kb)	-0.5774	1.3523
†Calcite (6kb)	-0.5837	1.3364
Ru	-0.0102	0.3222
†Sb	-1.3487	2.5814
†Bi (300°K)	-1.1851	1.0741
†Se (300°K)	13.8240	12,8409

† Materials have trigonal crystal structure.

$$W = \frac{1}{2} \; P^2(2S_{11}+2S_{12}+4S_{13}+S_{33})+2P(-\sigma_3{}')(S_{11}+S_{12}-S_{13}-S_{33})$$
$$+S_{11}(\sigma_1{}'^2+\sigma_2{}'^2+2\sigma_6{}'^2) +2S_{12}(\sigma_1{}'\sigma_2{}'-\sigma_6{}'^2) +2S_{13}\sigma_3{}'^2$$
$$+S_{33}\sigma_3{}'^2+S_{44}(\sigma_4{}'^2+\sigma_5{}'^2) \qquad (3.3.2)$$

The stored energy function cannot be split into deviatoric and hydrostatic parts alone unless the mixed term in the expression vanishes. This occurs in general only when $(S_{11}+S_{12}-S_{13}-S_{33}) = 0$, which is the condition of no deviatoric strain under pressure or no hydrostatic deformation under a deviatoric stress. (The stored energy per unit volume for cubic crystals can be split into deviatoric and hydrostatic parts).

In terms of strains, the stored energy per unit volume is given by

$$W = \frac{1}{2} [e]^T [C] [e] \qquad (3.3.3)$$

$$= \frac{1}{2} [e_1, e_2, e_3, e_4, e_5, e_6] \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

$$= \frac{1}{2}[e_1, e_2, e_3, e_4, e_5, e_6] \begin{bmatrix} C_{11}e_1+C_{12}e_2+C_{13}e_3 \\ C_{12}e_1+C_{11}e_2+C_{13}e_3 \\ C_{13}e_1+C_{13}e_2+C_{33}e_3 \\ C_{44}e_4 \\ C_{44}e_5 \\ \frac{1}{2}(C_{11}-C_{12})e_6 \end{bmatrix}$$

$$= \frac{1}{2} \; C_{11}(e_1{}^2+e_2{}^2+\frac{1}{2}e_6{}^2) + 2C_{12}(e_1e_2-\frac{1}{4}e_6{}^2) + 2C_{13}(e_1e_3+e_1e_3)$$
$$+C_{33}e_3{}^2+C_{44}(e_4{}^2+e_5{}^2)$$

Using $\underset{\sim}{e} = \underset{\sim}{e}' + \frac{1}{3}\,\text{tr}\,(\underset{\sim}{e})\,\underset{\sim}{I}$, where $\Delta V = \text{tr}\,(\underset{\sim}{e})$, this becomes

$$W = \frac{1}{2} (\Delta V/3)^2 (2C_{11}+2C_{12}+4C_{13}+C_{33}) \tag{3.3.4}$$

$$-2(\Delta V/3)^2 (e_3') (C_{11}+C_{12}-C_{13}-C_{33})$$

$$+C_{11}(e_1'^2+e_2'^2+\frac{1}{2}e_6^2) +2C_{12}(e_1'e_2'-\frac{1}{4}e_6^2)$$

$$-2C_{13}e_3'^2+C_{33}e_3'^2+C_{44}(e_4^2+e_5^2)$$

3.4 Incompressible Hexagonal Crystal

The incompressible hexagonal crystal is a model which leads to an interesting result. Incompressibility involves no volume change under any combination of stresses. This produces two conditions

$$2S_{11} +2S_{12} + 4S_{13} +S_{33} = 0 \tag{3.4.1}$$

$$S_{11} + S_{12} - S_{13} - S_{33} = 0 \tag{3.4.2}$$

Together, these conditions imply no deformation at all under hydrostatic stress and only deviatoric strain under deviatoric stress. Using the two conditions it is possible to eliminate two of the five stiffness parameters. Putting S_{13} and S_{33} in terms of S_{11} and S_{12}, the result is

$$S_{13} = -(S_{11} + S_{12}) \tag{3.4.3}$$

$$S_{33} = 2(3S_{11} + S_{12}) \tag{3.4.4}$$

Substituting these relations into the strain energy function, it becomes

$$W = \frac{1}{2} S_{11}(\sigma_1'^2+\sigma_2'^2+8\sigma_3'^2+2\sigma_6^2) + 2S_{12}(\sigma_1'\sigma_2'+2\sigma_3'^2-\sigma_6^2)$$

$$+ S_{44}(\sigma_4^2+\sigma_5^2) \tag{3.4.5}$$

The strain energy function for the incompressible hexagonal crystal is not pressure dependent. It is positive semi-definite in terms of the total stress and positive definite in terms of the deviatoric stress.

The reason for setting up the incompressible hexagonal crystal strain energy function is that a hypothetical analogy exists between this

form and the yield function. The yield function for isotropic materials corresponds to the form of the strain energy function for the incompressible, isotropic, cubic crystal, and the yield function for anisotropic materials set up by Hill {18} has the same form as the strain energy function for incompressible orthotropic crystals. If this analogy is extended to hexagonal crystals, then a yield function for materials with hexagonal symmetry would be of the form

$$2f(\sigma ij) = A'(\sigma_1'^2+\sigma_2'^2+8\sigma_3'^2+2\sigma_6^2) + B'(\sigma_1'\sigma_2'+2\sigma_3'^2-\sigma_6^2)$$

$$+C'(\sigma_4^2+\sigma_5^2) = 1 \qquad (3.4.6)$$

In terms of the actual normal stresses, this is

$$2f(\sigma ij) = A(\sigma_1^2+\sigma_2^2+10\sigma_3^2+8\sigma_1\sigma_2-10\sigma_1\sigma_3-10\sigma_2\sigma_3-6\sigma_6^2)$$

$$+2B(-3\sigma_3^2+3\sigma_1\sigma_3+3\sigma_2\sigma_3-3\sigma_1\sigma_2+3\sigma_6^2) \qquad (3.4.7)$$

$$+3C(\sigma_4^2+\sigma_5^2) = 1$$

where the stresses are referred to the principal axes of anisotropy of the element with the three direction parallel to the hexagonal axis.

The above yield function can be formed from the yield function

$$2f(\sigma ij) = F(\sigma y-\sigma z)^2+G(\sigma z-\sigma x)^2+H(\sigma x-\sigma y)^2+2L\tau yz^2+2M\tau zx^2+2N\tau xy^2=1$$

if the three supplementary conditions

$$G = H = A - F = (5A - 3B)/9 \qquad (3.4.8)$$

$$M = N = 3C/2 \qquad (3.4.9)$$

$$L = 2F + G \qquad (3.4.10)$$

are satisfied. These conditions would prevail at the instant of yielding of a material with hexagonal symmetry, with the x-axis parallel to the hexagonal axis, and are comparable to the isotropy conditions. If the yield function and potential function are assumed to be identical

and the plastic potential flow rule is used, then only certain modes of deformation are possible.

3.5 Implications of a Pressure Dependent Yield Function

A major assumption which is used repeatedly in the following work is that permanent plastic volume change cannot take place. In the range of pressure up to 100 kbar, Bridgeman {14}did not report any permanent plastic volume change for any single crystals which he studied in his work on the compressibility of solids. Further, the work of Alkins {15}, and Hanson and Wheeler {16} on deformation of poly crystalline aggregates showed that volume changes measured were small (<.5%) and most likely due to material inhomogeneities such as voids. They also measured extremely small volume increases after the initial decrease, which they took to be evidence of expansion of the grain boundaries in response to the mutual constraints imposed by the loading.

Consider a plastic body, assumed to be an aggregate of randomly oriented crystals, under the action of a hydrostatic pressure. As the pressure increases so too do the internal stresses. For an aggregate of cubic crystals, the internal stress field is a uniform hydrostatic pressure since no distortion of the individual crystals occurs under hydrostatic stress. For a random aggregate of hexagonal crystals however, the internal stress system cannot be hydrostatic since a hydrostatic stress produces distortion as well as volume change. If each grain were subjected to the same hydrostatic stress, the resulting distortion would be incompatible. In order that the individual grains fit together properly there must be internal shearing stresses in addition to the hydrostatic pressure. As the external hydrostatic pressure is increased, these shearing stresses eventually will cause plastic deform-

ation in some of the individual crystals. The plastic deformation is constrained to be of an elastic order of magnitude by the adjoining material which is not in a plastic state. Since plastic deformation is an irreversible process in the thermodynamic sense, there will be energy dissipation during the deformation. Because of this energy dissipation, there must be a net volume change in the cycle of application and removal of the pressure. Since there is no plastic volume change for single crystals, the volume change which occurs must be elastic. It is produced by the residual stresses between the crystals in the aggregate and is very small.

3.6 Conclusion

In work by Schmid and Boas {17} it has been shown that for most metal single crystals, including hexagonal crystals, yielding occurs when a critical shear stress is reached in the operative glide system. For cubic crystals the operative glide system is octahedral and yielding can be predicted using the von Mises yield criterion. (Recall that the von Mises yield criterion can be interpretted as implying yielding when the octahedral shear stress reaches a critical value.) It is purely speculation, but it may be possible that the incompressible crystal strain energy function used as a yield function implies yielding under critical shear in the operative glide system. If this does prove to be the case, the strain energy function (2.2.2) for anisotropic materials would correspond to the most general class of symmetry (orthotropic) and could be used safely for all materials no matter what the structure.

CHAPTER 4

Torsion of a Thin-Walled Tube

4.1 Introduction

Plastic torsion of a thin-walled tube produces axial strains if the tube is unconstrained and significant axial stresses if the tube is constrained axially. These effects can be due to the development of anisotropy in the material. Finite plastic strain is considered in this chapter; consequently the analysis is based on a rigid-plastic model. The wall thickness is assumed sufficiently small compared to the diameter that the deformation can be taken as uniform.

Hill {18} and Bailey et al {19} have studied the effects of anisotropy using a geometrical representation for the kinematics proposed by Hill, Figure 4.9, under the assumptions that:

(1) the principal axes of the deformation were near those of the isotropic case, implying that the effects of the axial and tangential strains on the kinematics could be ignored.

(2) the principal axes of anisotropy and deformation were coincident.

In the following work the kinematics of the problem are considered independently of the geometrical representation, using the polar decomposition of the deformation gradient to derive an expression for the orientation of the principal axes of the deformation which includes the effects of the axial and tangential strains. This

results in a more rigorous and elegant analysis.

The deformation is studied for the special cases of free and constrained ends and a general set of equations is developed to study the variation of the anisotropic parameters with deformation. The yield criterion and flow rule proposed by Hill {18} are used and the results are compared with an analysis of the same problem by Bailey et al {19} using the experimental data from that paper.

4.2 Kinematics of Deformation

The torsion of a thin-walled tube can be considered kinematically as an isochoric deformation equivalent to a uniform extension followed by simple shear. Referring to figures 4.1 (a), and (b), the deformation is defined by the mappings

$$\chi_1 = X_1\lambda_t$$

$$\chi_2 = X_2\lambda_a$$

$$\chi_3 = X_3\lambda_z$$

for the uniform extension

$\underset{\sim}{\chi} = (\chi_1,\chi_2,\chi_3) = (X_1,X_2.X_3)$

and

$$x_1 = \chi_1+K\chi_2$$

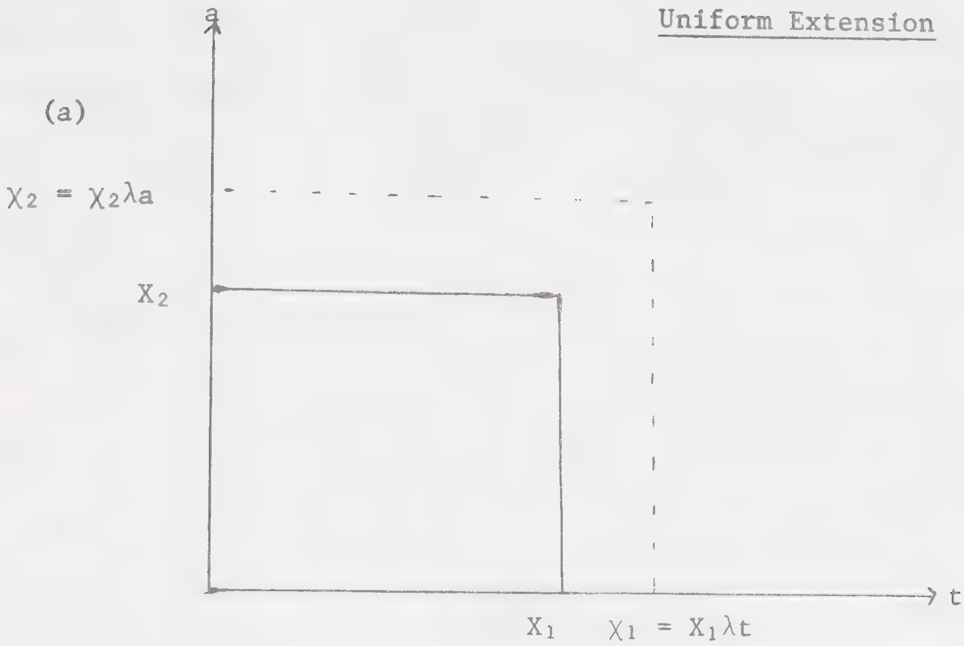
$$x_2 = \chi_2$$

$$x_3 = \chi_3$$

for the simple shear

$\underset{\sim}{x} = (x_1,x_2,x_3) = \underset{\sim}{x}(\chi_1,\chi_2,\chi_3)$

Note that $K = \tan\psi$



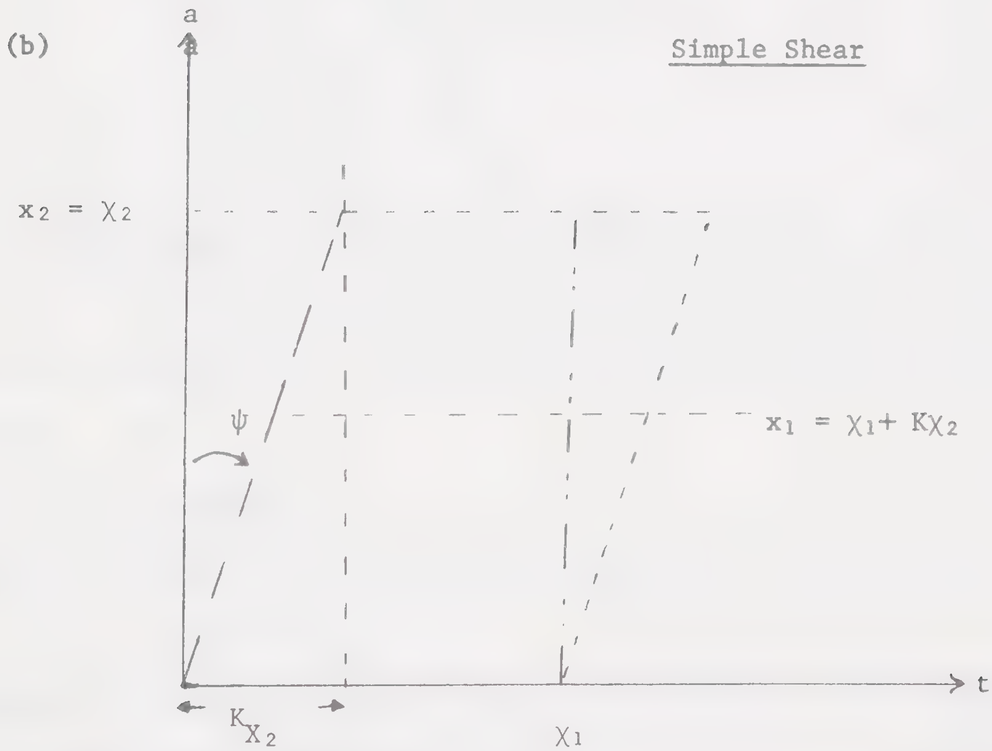


FIGURE 4.1

A Diagram of the Geometry of Uniform Extension and Simple Shear.

where $\lambda t = 1+\varepsilon t, \lambda a = 1 + \varepsilon a, \lambda z = 1 + \varepsilon z$

The incompressibility condition implies that $\lambda a \lambda t \lambda_z = 1$. Note also that the torsion strain is defined on the basis of the deformed length of the cylinder.

4.3 Deformation Gradient

The deformation gradient $\underset{\sim}{F}$ is given by the product of the deformation gradients of each of the simple cases. The matrix of components of the deformation gradient is given by

$$\{\underset{\sim}{F}\} = \{\underset{\sim b}{f}\} \cdot \{\underset{\sim a}{f}\} = \left\{\frac{\partial x}{\partial \underset{\sim}{\chi}}\right\}\left\{\frac{\partial \chi}{\partial \underset{\sim}{X}}\right\}$$

$$= \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda t & 0 & 0 \\ 0 & \lambda a & 0 \\ 0 & 0 & \lambda z \end{bmatrix} \tag{4.3.1}$$

$$= \begin{bmatrix} \lambda t & K\lambda a & 0 \\ 0 & \lambda a & 0 \\ 0 & 0 & \lambda z \end{bmatrix}$$

4.4 Polar Decomposition of the Deformation Gradient

The deformation gradient can be decomposed into the product of an orthogonal tensor and a symmetric positive definite tensor in two ways:

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \tag{4.4.1}$$

where $\underset{\sim}{U} = (\underset{\sim}{F}^T \underset{\sim}{F})^{\frac{1}{2}}$ is the pure stretch followed by a pure rotation.

$$\underset{\sim}{F} = \underset{\sim}{V}\,\underset{\sim}{R} \tag{4.4.2}$$

where $\underset{\sim}{V} = (\underset{\sim}{F}\underset{\sim}{F}^T)^{\frac{1}{2}}$ is the pure stretch which follows a pure rotation.

The form of the rotation matrix for a rotation about the three axis is

$$\{\underset{\sim}{R}\} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the angle α is the angle of rotation of the axes measured positive in the counterclockwise sense. The rotation matrix is taken to have the two dimensional form given, since one principal axis remains perpendicular to the wall of the tube. The first decomposition is

$$\{\underset{\sim}{F}\} = \{\underset{\sim}{R}\}\ \{\underset{\sim}{U}\} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{12} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda t & K\lambda a & 0 \\ 0 & \lambda a & 0 \\ 0 & 0 & \lambda z \end{bmatrix}$$

This is a matrix representation for the nine equations

$$U_{11}\cos\alpha - U_{21}\sin\alpha = \lambda t \qquad (4.4.3)$$

$$U_{12}\cos\alpha - U_{22}\sin\alpha = K\lambda a \qquad (4.4.4)$$

$$U_{13}\cos\alpha - U_{23}\sin\alpha = 0 \qquad (4.4.5)$$

$$U_{11}\sin\alpha + U_{21}\cos\alpha = 0 \qquad (4.4.6)$$

$$U_{12}\sin\alpha + U_{22}\cos\alpha = \lambda a \qquad (4.4.7)$$

$$U_{13}\sin\alpha + U_{23}\cos\alpha = 0 \qquad (4.4.8)$$

$$U_{31} = 0 \qquad (4.4.9)$$

$$U_{32} = 0 \qquad (4.4.10)$$

$$U_{33} = \lambda z \qquad (4.4.11)$$

Since $\underset{\sim}{U}$ is symmetric

$$U_{13} = U_{31} = 0 \quad \text{and} \quad U_{23} = U_{32} = 0$$

From (4.4.6) $U_{21} = U_{11}\tan\alpha$.

Substituting in (4.4.3) gives $U_{11} = \lambda t\cos\alpha$

which implies $U_{21} = -\lambda t\sin\alpha$.

From (4.4.7) $U_{12} = (\lambda a - U_{22}\cos\alpha)/\sin\alpha$

Substituting in (4.4.4) gives $U_{22} = \lambda a\,(\cos\alpha - K\sin\alpha)$

which implies $U_{12} = \lambda a(\sin\alpha + K\cos\alpha)$.

Since $U_{12} = U_{21}$, $\lambda a(\sin\alpha + K\cos\alpha) = -\lambda t\sin\alpha$

which implies that $\tan\alpha = -K\lambda a/(\lambda a + \lambda t)$. (4.4.12)

Consequently

$$\underset{\sim}{U} = \frac{1}{\sqrt{(\lambda a + \lambda t)^2 + (K\lambda a)^2}} \begin{bmatrix} \lambda t(\lambda a + \lambda t) & K\lambda a\lambda t & 0 \\ K\lambda a\lambda t & \lambda a(\lambda a + \lambda t) + (K\lambda a)^2 & 0 \\ 0 & 0 & \lambda z\sqrt{(\lambda a + \lambda t)^2 + (K\lambda a)^2} \end{bmatrix}$$

Similarly, the second polar decomposition gives

$$\{\underset{\sim}{F}\} = \{\underset{\sim}{V}\}\{\underset{\sim}{R}\} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda t & K\lambda a & 0 \\ 0 & \lambda a & 0 \\ 0 & 0 & \lambda z \end{bmatrix}$$

which is a matrix representation of the nine equations

$$V_{11}\cos\alpha + V_{12}\sin\alpha = \lambda t \quad (4.4.13)$$

$$-V_{11}\sin\alpha + V_{12}\cos\alpha = K\lambda a \quad (4.4.14)$$

$$V_{13} = 0 \quad (4.4.15)$$

$$V_{21}\cos\alpha + V_{22}\sin\alpha = 0 \quad (4.4.16)$$

$$-V_{21}\sin\alpha + V_{22}\cos\alpha = \lambda a \quad (4.4.17)$$

$$V_{23} = 0 \quad (4.4.18)$$

$$V_{31}\cos\alpha + V_{32}\sin\alpha = 0 \quad (4.4.19)$$

$$-V_{31}\sin\alpha + V_{32}\cos\alpha = 0 \quad (4.4.20)$$

$$V_{33} = \lambda z$$

Since $\underset{\sim}{V}$ is symmetric

$$V_{13} = V_{31} = 0 \quad \text{and} \quad V_{23} = V_{32} = 0$$

The remainder of the solution is

$$V_{11} = \lambda t\cos\alpha - K\lambda a\sin\alpha$$

$$V_{12} = \lambda t\sin\alpha + K\lambda a\cos\alpha$$

$$V_{21} = \lambda a\sin\alpha$$

$$V_{22} = \lambda a\cos\alpha$$

$$V_{33} = \lambda z$$

Setting $V_{12} = V_{21}$ gives $\tan\alpha = -K\lambda a/(\lambda a + \lambda t)$ (4.4.22)

Consequently

$$\{\underset{\sim}{V}\} = \frac{1}{\sqrt{(\lambda a+\lambda t)^2 + (K\lambda a)^2}} \begin{bmatrix} \lambda t(\lambda a+\lambda t) + (K\lambda a)^2 & K\lambda a^2 & 0 \\ K\lambda a & \lambda a(\lambda a+\lambda t) & 0 \\ 0 & 0 & \lambda z(\lambda a+\lambda t)^2+(K\lambda a)^2 \end{bmatrix}$$

$$\{\underset{\sim}{R}\} = \frac{1}{\sqrt{(\lambda a+\lambda t)^2 + (K\lambda a)^2}} \begin{bmatrix} (\lambda a+\lambda t) & K\lambda a & 0 \\ -K\lambda a & (\lambda a+\lambda t) & 0 \\ 0 & 0 & \sqrt{(\lambda a+\lambda t)^2 + (K\lambda a)^2} \end{bmatrix}$$

When $\lambda a = \lambda z = 1$, which is the simple shear case, $\tan\alpha = -K/2$

$$\{\underset{\sim}{R}\} = \frac{1}{\sqrt{4 + K^2}} \begin{bmatrix} 2 & K & 0 \\ -K & 2 & 0 \\ 0 & 0 & \sqrt{4 + K^2} \end{bmatrix}$$

$$\{\underset{\sim}{U}\} = \frac{1}{\sqrt{4 + K^2}} \begin{bmatrix} 2 & K & 0 \\ K & 2+K^2 & 0 \\ 0 & 0 & \sqrt{4 + K^2} \end{bmatrix}$$

$$\{\underset{\sim}{V}\} = \frac{1}{\sqrt{4 + K^2}} \begin{bmatrix} 2+K^2 & K & 0 \\ K & 2 & 0 \\ 0 & 0 & \sqrt{4 + K^2} \end{bmatrix}$$

(These relations for simple shear are easily verified). Henceforth, let

$$B = \sqrt{(\lambda a + \lambda t)^2 + (K\lambda a)^2}$$

4.5 Principal Axes of $\{\underset{\sim}{U}\}$

The eigenvalues of $\underset{\sim}{U}$ are the roots (η_1, η_2, η_3) of

$$\det\begin{bmatrix} \lambda t\ (\lambda a+\lambda t) - \eta & K\lambda a\lambda t & 0 \\ K\lambda a\lambda t & \lambda a(\lambda a+\lambda t) + (K\lambda a)^2-\eta & 0 \\ 0 & 0 & B\lambda z-\eta \end{bmatrix} = 0 \quad (4.5.1)$$

The third eigenvalue η_3 is determined easily by inspection to be

$$\eta_3 = B\lambda z \quad (4.5.2)$$

with the associated eigenvector

(0, 0, 1).

The other two eigenvalues are the roots of

$$\eta^2 - \{(\lambda a+\lambda t)^2 + (K\lambda a)^2\}\eta + \{\lambda a\lambda t(\lambda a+\lambda t) + \lambda a\lambda t\ (K\lambda a)\}^2 = 0$$
(4.5.2)

which are $\eta_1, \eta_2, = \{(\lambda a+\lambda t)^2+(K\lambda a)^2 \pm \sqrt{(\lambda a^2-\lambda t^2)^2+2(\lambda a^2+\lambda t^2)\ (K\lambda a)^2+\ (K\lambda a)^4}\}/2$

Let $C = \sqrt{(\lambda a^2-\lambda^2 t)+2(\lambda a^2+\lambda^2 t)(K\lambda a)^2+(K\lambda a)^4}$ (4.5.3)

Then

$$\eta_1 = (B^2+C)/2 \qquad \text{and } \eta_2 (B^2-C)/2$$

The principal axes of $\underset{\sim}{U}$ defined by the eigenvectors are the principal axes of stretch before rotation.

For $\eta = \eta_1$, the associated eigenvector is

$$(-\{\lambda a^2-\lambda t^2)+(K\lambda a)^2-C\},\ 2K\lambda a\lambda t,\ 0)$$

For $\eta = \eta_2$, the associated eigenvector is

$$(-\{\lambda a^2-\lambda t^2)+(K\lambda a)^2+C\},\ 2K\lambda a\lambda t,\ 0)$$

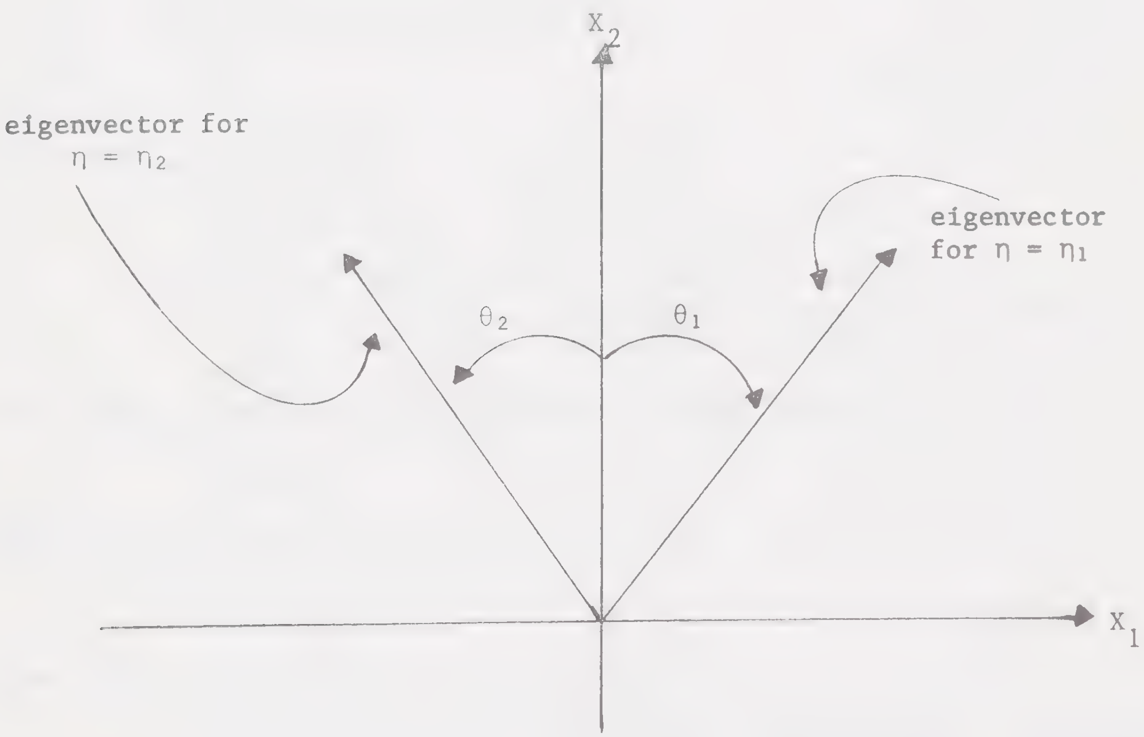


Figure 4.2

Orientation of the Principal Axes of $\{\underset{\sim}{U}\}$

Referring to Figure 4.2.

$$\theta_1 = \tan^{-1}\left(\frac{X_1}{X_2}\right)_{I} = \tan^{-1} -(\lambda a^2-\lambda t^2+(K\lambda a)^2-C)/2K\lambda a\lambda t \quad (4.5.4)$$

$$\theta_2 = \tan^{-1}\left(\frac{X_1}{X_2}\right)_{II} = \tan^{-1} (\lambda a^2-\lambda t^2+(K\lambda a)^2+C)/2K\lambda a\lambda t \quad (4.5.5)$$

$$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} = \frac{C/K\lambda a\lambda t}{0} = \infty$$

Therefore

$$\theta_1 + \theta_2 = \pi/2 \quad (4.5.6)$$

When $\lambda a = \lambda t = \lambda z = 1$

$$\theta_1 = \tan^{-1} (-K+\sqrt{4+K^2})/2 \quad (4.5.7)$$

$$\theta_2 = \tan^{-1} (K+\sqrt{4+K^2})/2 \quad (4.5.8)$$

4.6 Principal Axes of $\underset{\sim}{V}$

The eigenvalues of $\underset{\sim}{V}$ are the roots (η_1,η_2,η_3) of

$$\det \begin{bmatrix} \lambda t(\lambda a+\lambda t)+(K\lambda a)^2-\eta & K\lambda a^2 & 0 \\ K\lambda a^2 & \lambda a(\lambda a+\lambda t)-\eta & 0 \\ 0 & 0 & B\lambda z-\eta \end{bmatrix} = 0 \quad (4.6.1)$$

The third eigenvalue η_3 is determined by inspection to be

$$\eta_3 = B\lambda z$$

with the associated eigenvector

(0, 0, 1)

The other two eigenvalues are the roots of

$$\eta^2-\{(\lambda a+\lambda t)^2+(K\lambda a)^2\}\eta+\{\lambda a\lambda t(\lambda a+\lambda t)^2+(\lambda a\lambda t)(K\lambda a)^2\} = 0 \quad (4.6.2)$$

which are

$$\eta_1,\eta_2 = \{(\lambda a+\lambda t)^2+(K\lambda a)^2 \pm \sqrt{(\lambda a^2-\lambda t^2)^2+2(\lambda^2 a+\lambda^2 t)(K\lambda a)^2+K\lambda a)^4}\}/2 \quad (4.6.3)$$

Then

$$\eta_1 = (B^2+C)/2 \qquad \text{and } \eta_2 \; (B^2-C)/2$$

The principal axes of $\underset{\sim}{V}$ defined by the directions of the associated eigenvectors are coincident with the principal axes of the deformation. For $\eta = \eta_1$, the eigenvector is

$$(-\{\lambda a^2-\lambda t^2-(K\lambda a)^2-C\}, \;\; 2K\lambda a^2, \; 0)$$

For $\eta = \eta_2$, the eigenvector is

$$(-\{\lambda a^2-\lambda t^2-(K\lambda a)^2+C\}, \;\; 2K\lambda a^2, \; 0)$$

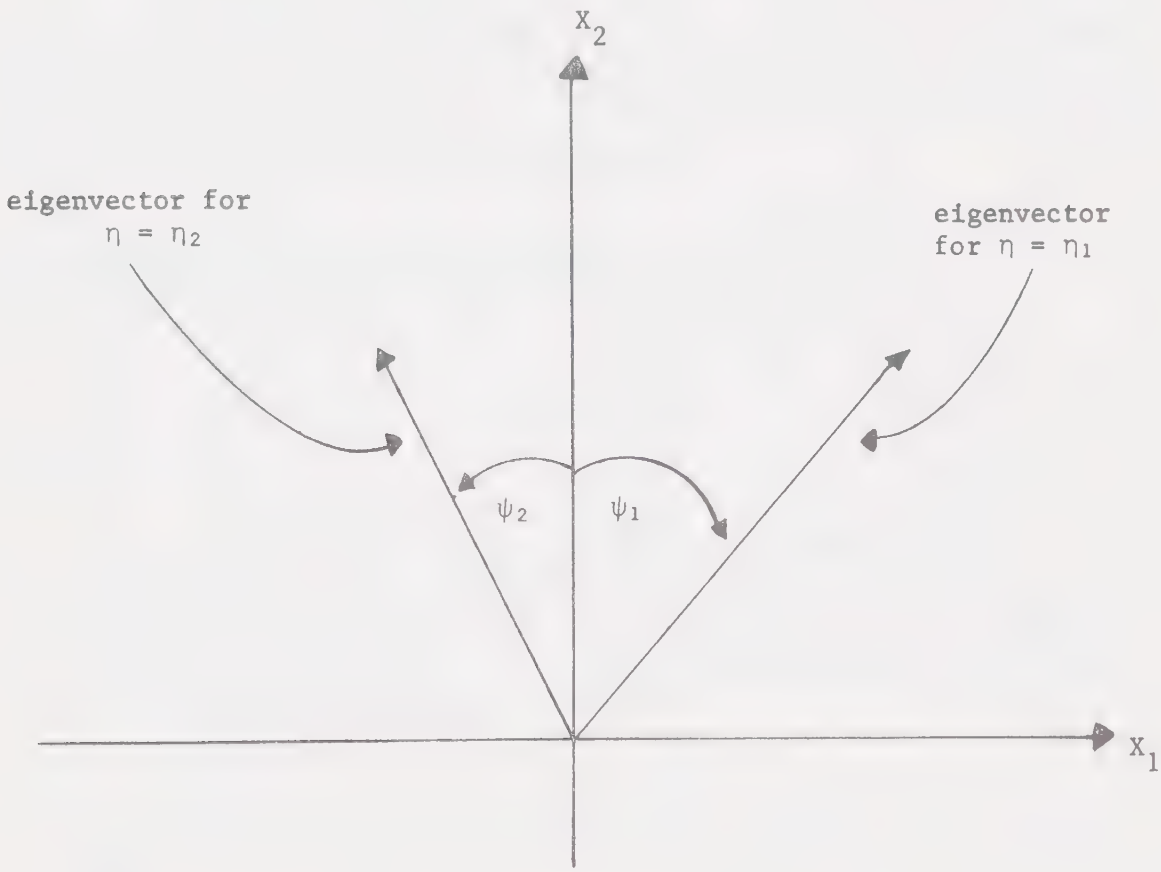


Figure 4.3

Orientation of the Principal Axes of $\underset{\sim}{V}$

Referring to Figure 4.3.

$$\psi_1 = \tan^{-1}\left(\frac{X_1}{X_2}\right)_{\mathrm{I}} = \tan^{-1} -(\lambda a^2 - \lambda t^2 - (K\lambda a)^2 - C)/2K\lambda a^2 \qquad (4.6.4)$$

$$\psi_2 = \tan^{-1}\left(\frac{X_1}{X_2}\right)_{\mathrm{II}} = \tan^{-1}(\lambda a^2 - \lambda t^2 - (K\lambda a)^2 + C)/2K\lambda a^2 \qquad (4.6.5)$$

$$\tan(\psi_1 + \psi_2) = \infty$$

Therefore, $\psi_1 + \psi_2 = \pi/2$ (4.6.6)

When $\lambda a = \lambda t = \lambda z = 1$

$$\psi_1 = \tan^{-1}\{K + \sqrt{4+K^2}\}/2 \qquad (4.6.7)$$

$$\psi_2 = \tan^{-1}\{-K + \sqrt{4+K^2}\}/2 \qquad (4.6.8)$$

Henceforth, $\phi = \psi_1$ in order to coincide with previous work by Hill {20} and Bailey et al {21} on the same problem.

4.7 Relation Between the Eigenvectors of $\underset{\sim}{U}$ and $\underset{\sim}{V}$

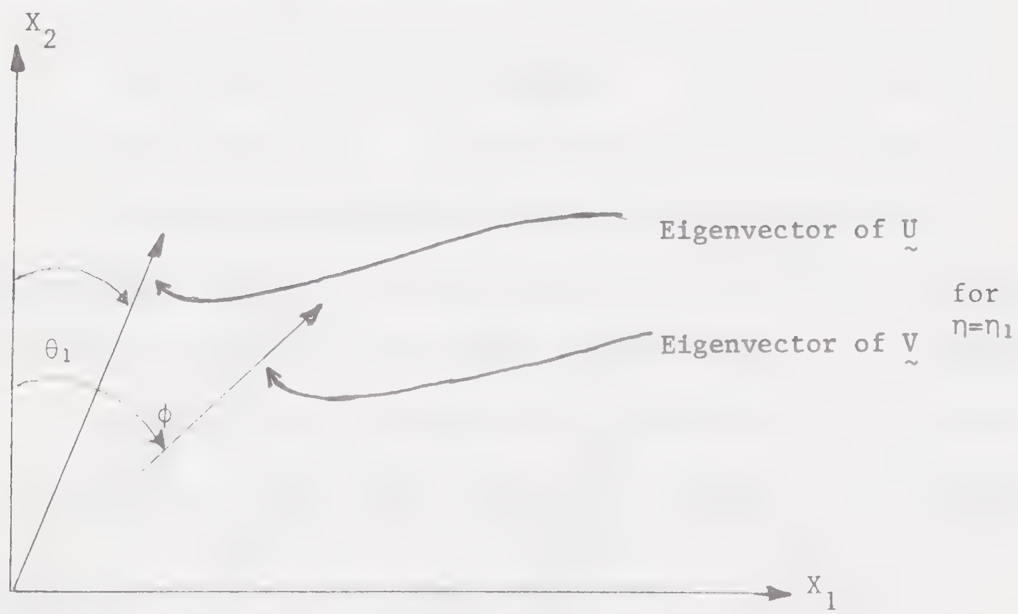


Figure 4.4

Relation of the Eigenvectors of $\underset{\sim}{U}$ and $\underset{\sim}{V}$

Referring to Figure 4.4 and the values of the various eigenvectors

$$\tan(\phi - \theta_1) = \frac{\tan\phi - \tan\theta_1}{1 + \tan\phi \tan\theta_1}$$

$$= \frac{K\lambda a}{2} \frac{(\lambda a-\lambda t)(\lambda a^2-\lambda t^2-C)+(\lambda a+\lambda t)(K\lambda a)^2}{(K\lambda a)^2\lambda a\lambda t+(\lambda a^2-\lambda t^2-C)^2-(K\lambda a)^4)/4}$$

$$= K\lambda a \frac{(\lambda a-\lambda t)(\lambda a^2-\lambda t^2-C)+(\lambda a+\lambda t)(K\lambda a)^2}{(K\lambda a)^2(\lambda a\lambda t)^2+(\lambda a^2-\lambda t^2)^2-(\lambda a^2-\lambda t^2)C}$$

$$= \frac{K\lambda a}{\lambda a+\lambda t}$$

$$= -\tan\alpha = \tan(-\alpha) \qquad (4.7.1)$$

Therefore, the angle between the eigenvectors of $\underset{\sim}{U}$ and $\underset{\sim}{V}$ is α, the angle of rotation in the polar decomposition of the deformation gradient.

When $\lambda a = \lambda t = \lambda z = 1$.

$$\mathrm{Tan}(\phi - \theta_1) = \frac{K}{2} = \tan(-\alpha)$$

4.8 Tensor Transformation Law for Stresses and Strains

The tensor transformation law in tensor form is

$$\varepsilon'\alpha\beta = \ell\alpha i\ell\beta j\varepsilon ij, \sigma'\alpha\beta = \ell\alpha i\ell\beta j\sigma ij$$

where $\ell\alpha o = \cos(x\alpha \,\hat{o}\, xi)$ (i.e., the cosine of the angle between the new and old directions measured +ve counterclockwise.)

Expanding these relations for the rotation of axes in Figure 4.5 gives

$$d\varepsilon x = \sin^2\phi\, d\varepsilon a + \cos^2\phi\, d\varepsilon t - \frac{1}{2} d\gamma \sin 2\phi \qquad (4.8.1)$$

$$d\varepsilon y = \cos^2\phi\, d\varepsilon a + \sin^2\phi\, d\varepsilon t + \frac{1}{2} d\gamma \sin 2\phi \qquad (4.8.2)$$

$$d\varepsilon xy = -\frac{1}{2}(d\varepsilon a - d\varepsilon t) \sin 2\phi + \frac{d\gamma}{2} \cos 2\phi \qquad (4.8.3)$$

where $d\gamma = 2d\gamma at$ ($d\gamma$ = engineering strain.) and

$$\sigma x = \sin^2\phi\, \sigma a + \cos^2\phi\, \sigma t - \tau at \sin 2\phi \qquad (4.8.4)$$

$$\sigma y = \cos^2\phi\, \sigma a + \sin^2\phi\, \sigma t + \tau at \sin 2\phi \qquad (4.8.5)$$

$$\sigma xy = -\frac{1}{2}(\sigma a - \sigma t) \sin 2\phi + \tau at \cos 2\phi \qquad (4.8.6)$$

The angle ϕ is measured positively in the clockwise direction.

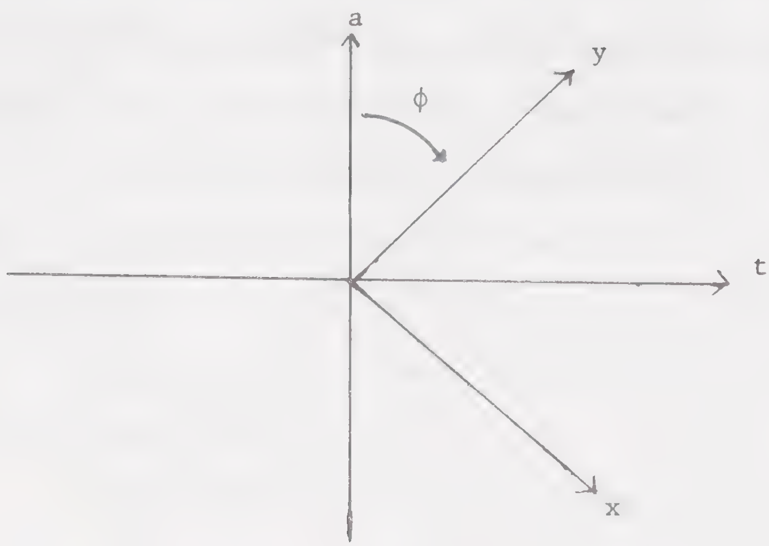


Figure 4.5

Orientation of the (x,y) Anisotropic Axes to the (t,a) Major Axes

4.9 Variation of the Anisotropic Parameters During Deformation

As the thin-walled tube is plastically strained, the initially isotropic material gradually develops anisotropy. The yield criterion given by Hill

$$2f(\sigma ij) = F(\sigma y-\sigma z)^2 + G(\sigma z-\sigma x)^2 + H(\sigma x-\sigma y)^2 + 2L\tau yz + 2M\tau zx^2$$
$$+ 2N\tau xy^2 = 1$$

is assumed to hold. In order to describe fully the state of anisotropy in the wall of the tube it is necessary to know the orientation of the principal axes of anisotropy and the six anisotropic parameters (or the six yield stresses). It is assumed that the principal axes of anisotropy coincide at each instant with the principal axes of the deformation as determined by the angle ϕ.

Since τyz and τzx are zero for the torsion of a thin-walled tube about the y-axis, the parameters L and M have no effect on the

behaviour of the yield function for this problem. The only way that L and M can be evaluated is by direct measurement of the shear yield stresses. This is very difficult, if not impossible, for the thin wall. It is assumed henceforth that the values of L, M and N are equal due to latent hardening of the material leaving four parameters to evaluate.

Since $\sigma z = \sigma t = 0$ for the loading in the problem, the stress-strain increment relations are

$$d\varepsilon x = d\lambda\{H(\sigma x-\sigma y) + G\sigma x\} = \sin^2\phi d\varepsilon a + \cos^2\phi d\varepsilon t - \frac{d\gamma}{2} \sin 2\phi \qquad (4.9.1)$$

$$d\varepsilon y = d\lambda\{H(\sigma y-\sigma x) + F\sigma y\} = \cos^2\phi d\varepsilon a + \sin^2\phi d\varepsilon t + \frac{d\gamma}{2}\sin 2\phi \qquad (4.9.2)$$

$$d\varepsilon z = d\lambda\{G\sigma x + F\sigma y\} = -(d\varepsilon x + d\varepsilon y) = -(d\varepsilon a + d\varepsilon t) \qquad (4.9.3)$$

$$d\varepsilon xy = d\lambda N\tau xy =\{-(d\varepsilon a-d\varepsilon t) \sin^2\phi + d\gamma\cos 2\phi\}/2 \qquad (4.9.4)$$

These relations give three independent equations for the four unknowns. Since $d\lambda = \tau d\gamma$ the stress-strain increment relations become

$$H(\sigma x-\sigma y) + G\sigma x = (\sin^2\phi d\varepsilon a+\cos^2\phi d\varepsilon t-\frac{d\gamma}{2}\sin 2\phi)/\tau d\gamma \qquad (4.9.5)$$

$$H(\sigma y-\sigma x) + F\sigma y = (\cos^2\phi d\varepsilon a+\sin^2\phi d\varepsilon t+\frac{d\gamma}{2}\sin 2\phi)/\tau d\gamma \qquad (4.9.6)$$

$$G\sigma x + F\sigma y = (d\varepsilon a + d\varepsilon t)/\tau d\gamma \qquad (4.9.7)$$

$$N\tau xy = \{-(d\varepsilon a - d\varepsilon t) \sin 2\phi + d\gamma\cos 2\phi\}/2\tau d\gamma \qquad (4.9.8)$$

One further equation can be obtained from the definitions of the anisotropic parameters in terms of the yield stresses.

$$G + H = \frac{1}{X^2}, \quad H + F = \frac{1}{Y^2}, \quad F + G = \frac{1}{Z^2} \qquad (4.9.9)$$

In order to obtain explicit relations for the variation of the anisotropic parameters using the equations developed, it is necessary to have measurements of the response to the loading. For the tor-

sional loading $\sigma z = \sigma t = 0$. It is necessary to measure two of εa, εt, εz as functions of γ. It is also required that one of X, Y, Z must be known. Usually εa and εt are the easiest strains to measure and Z is the only yield stress which can be measured without destroying the specimen. (By measuring the hardness of material perpendicular to the wall.) It is also necessary to measure the shearing torque and the axial load (if the tube is constrained) as functions of γ. The system of equations used for the remainder of the analysis is

$$G\sigma x + H(\sigma x-\sigma y) = (\tau^2 o/\tau d\gamma)(\sin^2\phi d\varepsilon a+\cos^2\phi d\varepsilon t-\frac{d\gamma}{2}\sin 2\phi) \quad (4.9.10)$$

$$F\sigma y + G\sigma x = (\tau^2 o/\tau d\gamma)(d\varepsilon a + d\varepsilon t) \quad (4.9.11)$$

$$F + G = (\tau o/z)^2 \quad (4.9.12)$$

$$N\tau xy = (\tau^2 o/2\tau d\gamma)(-(d\varepsilon a-d\varepsilon t)\sin 2\phi + d\gamma\cos 2\phi) \quad (4.9.13)$$

where F, G, H, N are in the non-dimensional form with τo being equal to the yield shear stress at first yield. Also,

$$\phi = \tan^{-1}\frac{1}{2}\left\{\frac{\lambda t^2-\lambda a^2 + (K\lambda a)^2 + C}{K\lambda a^2}\right\}$$

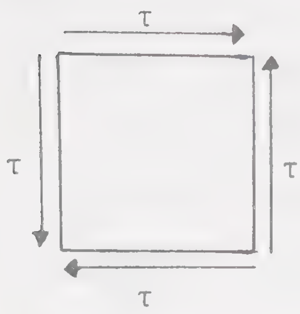
where $C = \sqrt{(\lambda a^2-\lambda t^2)^2 + 2(\lambda a^2+\lambda t^2)(K\lambda a)^2 + (K\lambda a)^4}$

If ϕ is assumed not to vary too much from the case where $\lambda a = \lambda t = \lambda z = 1$, then

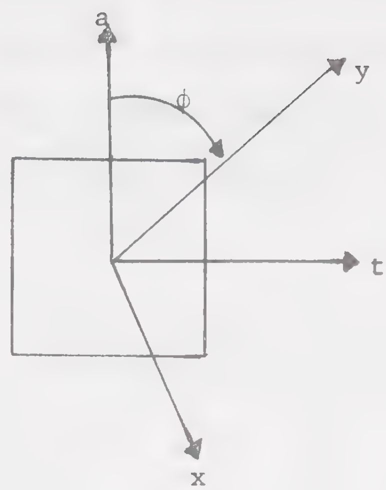
$$\phi = \tan^{-1}\frac{1}{2}\{K + \sqrt{4+K^2}\} = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{K}{2}$$

The validity of the assumption may be checked by direct comparison with full form for ϕ given above.

4.10 Tube Not Constrained Axially



a) stress system



b) orientation of principal axes

Figure 4.6

Stress System and Orientation of the Principal Axes of Anisotropy For the Unconstrained Tube

The only non-zero stress is $\tau at = \tau ta = \tau$. The stress system referred to the principal axes of anisotropy is

$$-\sigma x = \sigma y = \tau \sin 2\phi$$

$$\tau xy = \tau \cos 2\phi \qquad (4.10.1)$$

The yield function (2.2.2) using these stresses becomes

$$2f(\sigma ij) = F\sigma y^2 + G\sigma x^2 + H(\sigma x - \sigma y)^2 + 2N\tau xy^2 = 1$$

$$= \tau^2 (\sin^2 2\phi (F + G + 4H - 2N)) = 1$$

implying

$$\tau = (2N + (F + G + 4H - 2N) \sin^2 2\phi)^{-\frac{1}{2}} \qquad (4.10.2)$$

The system of equations for the unconstrained tube is

$$G + 2H = -(\tau o/\tau)^2(\sin^2\phi \frac{d\varepsilon a}{d\gamma} + \cos^2\phi \frac{d\varepsilon a}{d\gamma} - \frac{1}{2}\sin 2\phi)/\sin 2\phi \qquad (4.10.3)$$

$$F - G = (\tau o/\tau)^2(d\varepsilon a/d\gamma + d\varepsilon t/d\gamma)/\sin 2\phi \qquad (4.10.4)$$

$$F + G = (\tau o/z)^2 \qquad (4.10.5)$$

$$N = (\tau o/\tau)^2 \{-(\frac{d\varepsilon a}{d\gamma} - \frac{d\varepsilon t}{d\gamma}) \tan 2\phi + 1\}/2 \qquad (4.10.6)$$

Since εa and εt are functions of γ only, the ratios $d\varepsilon a/d\gamma$ and $d\varepsilon t/d\gamma$ are equivalent to the derivatives. From the curves published by Bailey et al {21} the strains are given by the approximate functions

$$\varepsilon a(\gamma) = (.01)\ (.25\gamma + .171\gamma^2 + .025\gamma^3 + .00104\gamma^4) \quad (4.10.7)$$

$$\varepsilon t(\gamma) = -\varepsilon a(\gamma)/3$$

and the derivatives are given by

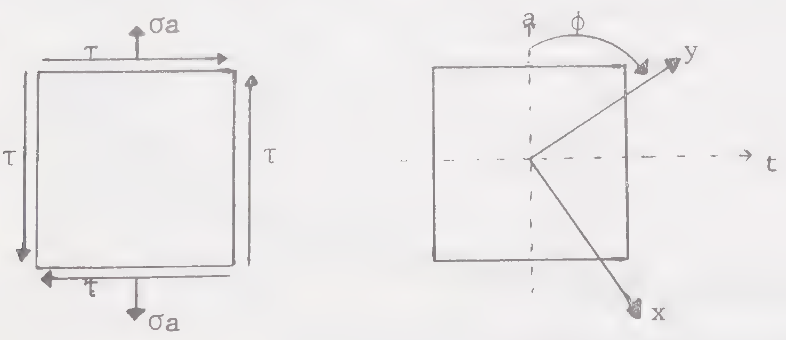
$$\frac{d\varepsilon a}{d\gamma}(\gamma) = (.01)\ (.25 + .342\gamma - .075\gamma^2 + .00416\gamma^3) \quad (4.10.8)$$

$$\frac{d\varepsilon t(\gamma)}{d\gamma} = -\frac{d\varepsilon a}{d\gamma}\ /3$$

These functions are fifth order polynomial expansions (note fifth order term = 0) and are not valid for $\gamma>10$ where the functions begin to diverge. The expressions for $\tau/\tau o$ and $z/\tau o$ proved to be difficult to model and the calculation of the variation of the anisotropic parameters was undertaken using point by point input of values derived from the paper by Bailey et al {21}. The results are presented in Figure 4.7.

4.11 Tube Constrained Axially

Though this case was not analyzed or experimented on by Bailey et al {21}, it was decided to include it for completeness. The stress system and orientation of the principal axes are shown in Figure 4.8.

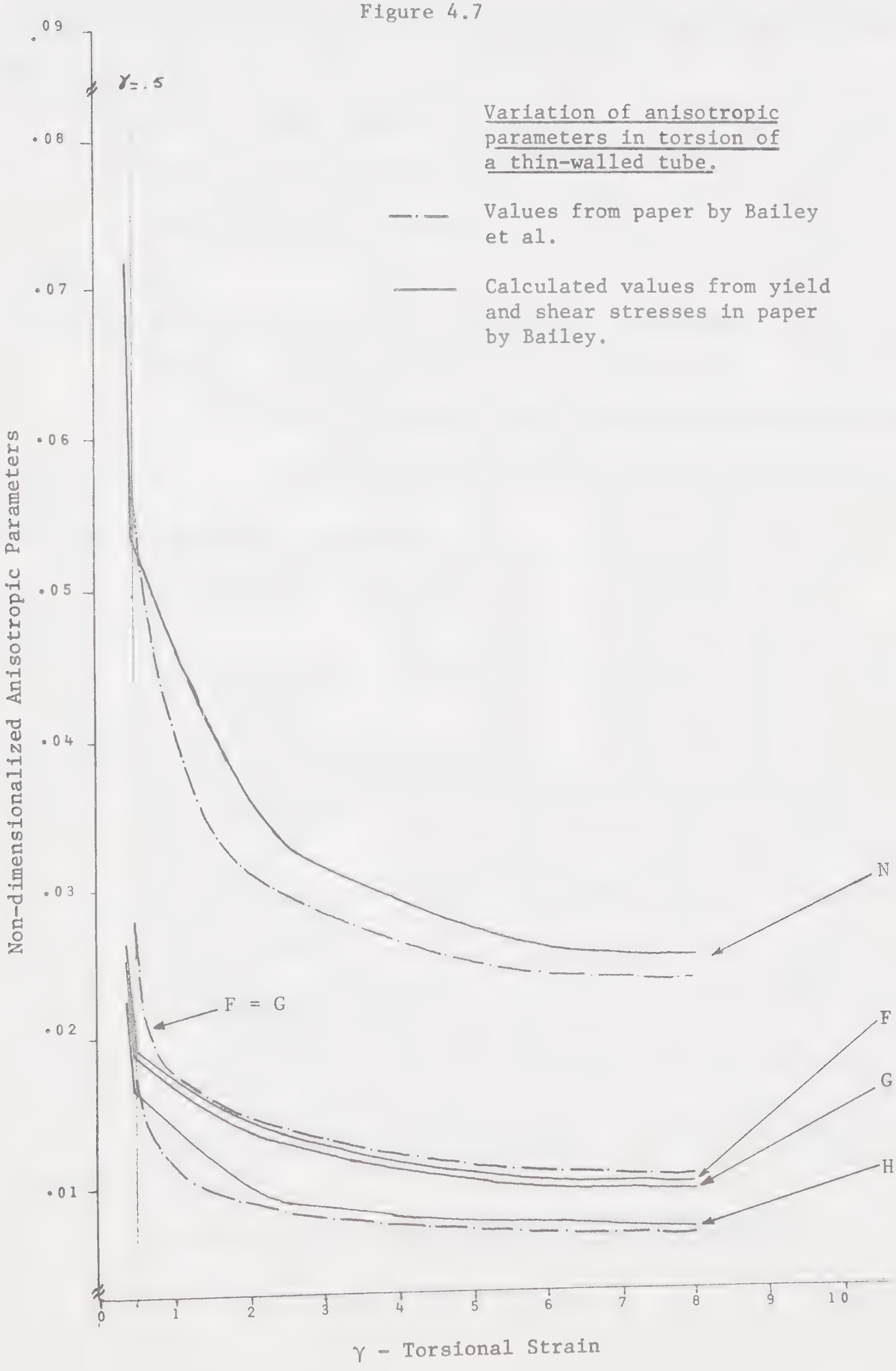


a) stress system b) Orientation of principal axes

Figure 4.8

Stress System and Orientation of Principal Axes of Anisotropy for the Constrained Tube

Figure 4.7



Variation of anisotropic parameters in torsion of a thin-walled tube.
Values from paper by Bailey et al.
Calculated values from yield and shear stresses in paper by Bailey.
Non-dimensionalized Anisotropic Parameters
γ - Torsional Strain
γ = .5
F = G
N
F
G
H
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
1
2
3
4
5
6
7
8
9
10

The stress system referred to the principal axes of anisotropy is given by

$$\sigma x = \tau(A \sin^2\phi - \sin 2\phi)$$

$$\sigma y = \tau(A \cos^2\phi + \sin 2\phi) \qquad (4.11.1)$$

$$\sigma xy = \tau(-\frac{A}{2} \sin 2\phi + \cos 2\phi)$$

where $A = \sigma a/\tau$

Under this stress system the yield criterion (2.2.2) becomes

$$2f(\sigma ij) = F\sigma y^2 + G\sigma x^2 + H(\sigma x-\sigma y)^2 + 2N\tau x^2 y = 1$$

$$= F(A \cos^2\phi+\sin 2\phi)^2+G(A\sin^2\phi-\sin 2\phi)^2+H(-A\cos 2\phi-2\sin 2\phi)^2$$

$$+ 2N (-\frac{A}{2} \sin 2\phi + \cos 2\phi)^2 = 1/\tau^2 \qquad (4.11.2)$$

Two special cases were considered

A) $d\varepsilon a = d\varepsilon t = d\varepsilon z \equiv 0$

This gives

$$\phi = \frac{\pi}{4} + \frac{1}{2} \tan^{-1}\frac{K}{2}$$

and the system of equations becomes

$$-G (A \sin^2\phi-\sin 2\phi) + H(A \cos 2\phi+ 2\sin 2\phi) = (\tau o/\tau)^2 \sin 2\phi/2 \qquad (4.11.3)$$

$$F (A \cos^2\phi+\sin 2\phi) + G (A \sin^2\phi- \sin 2\phi) = 0 \qquad (4.11.4)$$

$$F + G = (\tau o/z)^2 \qquad (4.11.5)$$

$$N = \frac{(\tau o/\tau)^2 \cos 2\phi}{2(-\frac{A}{2}\sin 2\phi+\cos 2\phi)} \qquad (4.11.6)$$

B) $d\varepsilon a = 0, \quad d\varepsilon t = -d\varepsilon z$

The stress system remains the same as in the above case. The angle ϕ is given by

$$\phi = \tan^{-1} \frac{1}{2}\left\{\frac{(\lambda t^2-1) + K^2 + \sqrt{(1-\lambda t^2)^2+2(1+\lambda t)^2K^2+K^4}}{K}\right\}$$

The system of equations is given by

$$-G(A \sin^2\phi - \sin 2\phi) + H(A \cos 2\phi + 2 \sin 2\phi)$$

$$= (\tau o/\tau)^2 \left(\frac{\sin 2\phi}{2} + \cos^2\phi \frac{d\varepsilon z}{d\gamma}\right) \quad (4.11.7)$$

$$F(A \cos^2\phi + \sin 2\phi) + G(A \sin^2\phi - \sin 2\phi)$$

$$= -(\tau o/\tau)^2 \frac{d\varepsilon z}{d\gamma} \quad (4.11.8)$$

$$F + G = \left(\frac{\tau o}{z}\right)^2 \quad (4.11.9)$$

$$N = \left(\frac{\tau o}{\tau}\right)^2 \frac{-\sin 2\phi \frac{d\varepsilon z}{d\gamma} + \cos 2\phi}{-A\sin 2\phi + 2\cos 2\phi} \quad (4.11.10)$$

There should be no difficulty involved in actually running an experiment to determine the variation of the anisotropic parameters. The only measurement not required before is one of axial thrust produced during the torsion. It would also be possible to do a complete analysis for arbitrary axial stress (as long as it was below the yield point) and torsion. (In effect Hohenemsers problem of combined tension and torsion could be analyzed for the variation of the anisotropic parameters.)

4.12 Conclusions

The paper by Bailey et al {21} which has been cited often here had much the right approach to the problem, but the analysis is incorrect. The error arises in the transformation equations used:

$$d\varepsilon x = \frac{1}{2}(d\varepsilon a + d\varepsilon t) + \frac{1}{2}(d\varepsilon a - d\varepsilon t) \cos 2\theta + \frac{1}{2} d\gamma \sin 2\theta$$

$$= d\varepsilon a \cos^2\theta + d\varepsilon t \sin^2\theta + \frac{d\gamma}{2} \sin 2\theta \quad (4.12.1)$$

$$d\varepsilon y = \frac{1}{2}(d\varepsilon a + d\varepsilon t) - \frac{1}{2}(d\varepsilon a - d\varepsilon t) \cos 2\theta - \frac{1}{2} d\gamma \sin 2\theta$$

$$= d\varepsilon a \sin^2\theta + d\varepsilon t \cos^2\theta - \frac{d\gamma}{2} \sin 2\theta \quad (4.12.2)$$

$$d\varepsilon xy = -\frac{1}{2}(d\varepsilon a - d\varepsilon t)\sin 2\theta + \frac{d\gamma}{2}\cos 2\theta \qquad (4.12.3)$$

where "θ is the angle of inclination of the (x, y) anisotropic axes to the (a, t) major axes, and is considered positive anticlockwise." (see Figure 4.5).

Although not explicitly stated, Bailey uses $\theta = -\phi$. The correct forms of the transformation equations are (4.8.1), (4.8.2), (4.8.3) (they can be verified by using the relations in appendix IV of Hill {22}). The error is due to the fact that the (a, t) axes are not a right-handed system of co-ordinates, and the correct forms can be produced if the a and t are interchanged in the transformation equations used by Bailey.

Also, the equations for ϕ and $d\phi/d\gamma$ are unnecessarily complicated; for the isotropic case

$$\phi = \sin^{-1}\left[\frac{\sin\psi/2}{(1 + (2\cos\psi\,\sin\psi/2) - 2\cos\psi)^{\frac{1}{2}}}\right] \qquad (4.12.4)$$

where $\tan\psi = 2/\gamma = 2/K$

$$\frac{d\phi}{d\gamma} = \left(\frac{-2}{\gamma^2+4}\right)\left[\frac{A^{-\frac{1}{2}}}{2}\{A-\sin^2\psi/2\}^{-\frac{1}{2}}\right] x \qquad (4.12.5)$$

$$x\left[A\cos\psi/2 - A^{\frac{1}{2}}\sin\psi/2\left\{\frac{4\sin\psi/2}{\tan\psi}\frac{(\tan\psi\cos\psi/2 - 2\sin\psi/2\sec^2\psi)}{\tan^2\psi}\right.\right.$$

$$\left.\left. + 2\sin\psi\right\}\right]$$

Where $A = \{1 + \left(\frac{2\sin\psi/2}{\tan\psi}\right)^2 - 2\cos\psi\}$

whereas the analysis using the polar decomposition gives (for the isotropic case)

$$\phi = \tan^{-1}\frac{1}{2}(\gamma + \sqrt{4+\gamma^2}) = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{\gamma}{2} \qquad (4.12.6)$$

and

$$\frac{d\phi}{d\gamma} = \frac{1}{\gamma^2+4} \qquad (4.12.7)$$

(It is possible to get the last relation for ϕ directly from the geometry of Figure 4.9.)

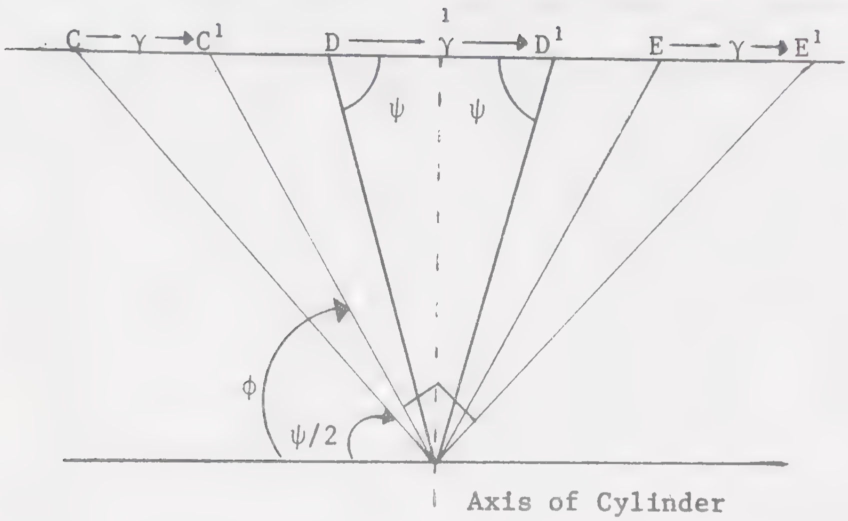


Figure 4.9

Geometry of Deformation for the Plastic Torsion of a Thin Walled Tube

Another fairly minor point is that it is totally unnecessary, and perhaps even harmful in a numerical sense, to put εa and εt as functions of ϕ. It is more natural to derive the strain functions in terms of γ and it is imperative that they be in that form if the correct form for ϕ is to be used

$$\phi = \tan^{-1}\frac{1}{2}\frac{\{\lambda t^2-\lambda a^2+(\gamma\lambda a)^2+\sqrt{(\lambda a^2-\lambda^2 t)^2+2(\lambda a^2+\lambda t^2)(\gamma\lambda a)^2+(\gamma\lambda a)^4}}{\gamma\lambda a^2}$$

For the case studied, the difference in the angles ϕ between the assumed isotropic form and the correct form above is the order of 10^{-3} radians and decreases for very large values of γ, showing that the assumption is valid. Referring to Fig. 4.7, the variation of the anisotropic parameters as calculated by the two analyses is presented. Differences as large as 40% in the value of F (at γ = .5) and 15% in the value of N

(between γ = 1.5 and 2.5) exist between the calculated values, although at large strains the differences are less than 5%. The differences are most probably due to the use of the incorrect transformation equations by Bailey, {19} , although round off and other numerical errors from the clumsy forms for ϕ , $\frac{d\phi}{d\gamma}$ and $\frac{d\varepsilon a}{d\gamma}$ and $\frac{d\varepsilon t}{d\gamma}$ may also have an effect.

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